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Inelastic scattering and interactions of three-wave parametric solitons

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We study the interactions of velocity-locked three-wave parametric solitons in a medium with quadratic nonlinearity and dispersion. We reveal that the inelastic scattering between three-wave solitons and linear waves may be described in terms of analytical solutions with dynamically varying group velocity, or boomerons. Moreover, we demonstrate the elastic nature of three-wave soliton-soliton collisions and interactions.

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Three-wave resonant interactions (TWRIs) are widespread in various branches of physics, as they describe the resonant mixing of waves in weakly nonlinear and dispersive media. The TWRI model is typically encountered in the description of any conservative nonlinear medium where the nonlinear dynamics can be considered as a perturbation of the linear wave solution, the lowest-order nonlinearity is quadratic in the field amplitudes, and the phase-matching (or resonance) condition is satisfied. Solutions for the TWRIs have been known for a long time [1–9], and extensive applications are found in nonlinear optics (parametric amplification, frequency conversion, stimulated Raman and Brillouin scattering), plasma physics (laser-plasma interactions, radio frequency heating, plasma instabilities), acoustics (light-acoustic interactions), fluid dynamics (interaction of water waves), and solid state physics (wave-wave scattering). Soliton solutions of the TWRIs are of particular interest in the study of coherent energy transport and frequency conversion. Indeed, the potential applicability of solitons originates from their particlelike behavior. In the context of TWRI solitons, their different component waves should propagate locked together as a single coherent structure, an effect that has no counterpart for linear waves [1,3–5,10–14]. We recently introduced such structures as a multiparameter TWRI soliton family, consisting of triplets of bright-bright-dark waves (or simultons) which travel with a common group velocity [15]. We could also identify the conditions for the stability and instability of these TWRI simultons (TWRISs) [15].

In this Rapid Communication we reveal and explore another feature of the particlelike nature of the TWRI simultons discussed in Refs. [1,3–5,15], namely, the inelastic scattering of TWRISs with particular linear waves. More specifically, in this work (i) we present a multiparameter family of explicit solutions (boomerons) of the three-wave equations which exactly describe the scattering of a TWRIS with a linear wave, (ii) we prove that this family includes as an asymptotic state as $t \to \pm\infty$ the TWRIS of Refs. [1,3–5,15], and (iii) we show that an unstable TWRIS decays to a stable TWRIS by properly accelerating its group velocity to a certain different value. We would like to emphasize that the inelastic scattering phenomenon which is investigated here is associated with the excitation (decay) of stable (unstable) simultons by means of the absorption (emission) of the energy carried by an isolated linear pulse. The decay (excitation) of simultons is associated with their speedup (slowing down) and creation of another triplet with complementary stability properties. The scattering of solitons and linear waves, which has been so far largely unexplored in the literature, is analytically described here by means of boomeron-type solutions. Such a soliton-wave scattering process is analogous to the interaction of radiation with a two-level atomic system: transitions among excited and ground soliton states are induced by the absorption and spontaneous emission of a proper linear wave. Lastly, we shall reveal the elastic nature of the TWRI soliton-soliton collisions and interactions.

The coupled partial differential equations that rule TWRIs in (1+1) dimensions read as [2]

$$E_{1t} - V_1 E_{1z} = E_2^* E_3^*,$$

$$E_{2t} - V_2 E_{2z} = - E_1^* E_3^*,$$

$$E_{3t} - V_3 E_{3z} = E_1^* E_2^*, \qquad (1)$$

where the subscripts t and z denote derivatives in the longitudinal and transverse dimensions, $E_n = E_n(z,t)$ are the complex wave amplitudes with velocities V_n, and $n=1,2,3$. We chose here $V_1 > V_2 > V_3$ which, together with the above choice of the signs before the quadratic terms, entails the nonexplosive character of the three-wave interaction [9]. In the following, with no loss of generality, we shall consider Eqs. (1) in a reference frame with $V_3=0$. Since we consider resonant interactions, the frequencies and momenta of the three waves must satisfy the prescriptions $\omega_1+\omega_2+\omega_3=0$ and $k_1+k_2+k_3=0$.

The TWRI equations (1) represent an infinite-dimensional Hamiltonian system, which conserves the Hamiltonian, the sum of the energies of waves E_1 and E_2, the sum of the energies of waves E_2 and E_3, and the total transverse momentum (see Ref. [15] for details).

Equations (1) exhibit a three-parameter family of simulton solutions in the form of bright-bright-dark triplets that travel with a common or locked velocity V [15]. The most

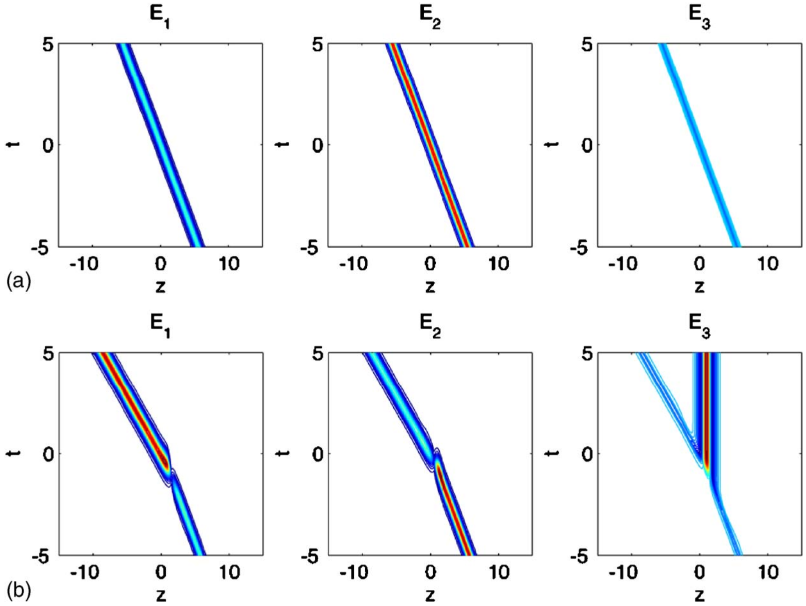

FIG. 1. (Color online) (a) Analytical solution and (b) numerical propagation of an unstable TWRIS, which coincides with a boomeron solution. Here $V_1=2, V_2=1$. The simulton velocity is $V=1.1$ ($V<V_{cr}\approx 1.3$).

remarkable physical property of these simultons is that their speed V may be continuously varied by means of adjusting the energy of the two bright pulses. The propagation stability analysis of TWRISs reveals that a triplet is no longer stable whenever its velocity V decreases below a well-defined (critical) value, namely, $V<V_{cr}=2V_1V_2/(V_1+V_2)$ [15]. As an example, in Fig. 1(a) we show the contour plot of the amplitude of the three waves that compose an unstable simulton. These plots should be compared with Fig. 1(b), obtained from numerical propagation with an initial (i.e., at $t=-5$) condition given by the exact simulton solution of Fig. 1(a). The results of Fig. 1 illustrate that an unstable simulton with $V<V_{cr}$ decays into a stable simulton with $V>V_{cr}$. This process is accompanied by the emission of an isolated pulse in wave E_3. It is quite remarkable that the simulton decay and wave emission as numerically observed in Fig. 1(b) may be exactly reproduced in terms of an analytical higher-order soliton solution with varying speed, or boomeron. Such a solution was found by means of the techniques described in Ref. [16], and it can be expressed as

$$E_1 = \frac{2pV_2}{\Delta}\sqrt{\frac{2V_1}{V_1-V_2}}e^{iq_1z_1}(H_+^*e^{-i\theta}-H_-^*e^{i\theta}), \tag{2a}$$

$$E_2 = \frac{2pV_1}{\Delta}\sqrt{\frac{2V_2}{V_1-V_2}}e^{iq_2z_2}[\sqrt{(1-Q)/(1+Q)}H_+e^{i(\beta+\theta)} - \sqrt{(1+Q)/(1-Q)}H_-e^{-i(\beta+\theta)}], \tag{2b}$$

$$E_3 = a\sqrt{V_1V_2}e^{iq_3z_3} - \frac{\Delta}{4p}\left(\frac{V_1-V_2}{V_1V_2}\right)E_1^*E_2^*, \tag{2c}$$

where

$$\Delta = 1 + \frac{|H_+|^2}{1+Q} + \frac{|H_-|^2}{1-Q} - 2\cos(\beta)\mathrm{Re}(H_+H_-^*e^{i(\beta+2\theta)}),$$

$$H_\pm(z,t) = e^{(-B_\pm+i\chi_\pm)z}e^{[-2V_1V_2/(V_1-V_2)](p-ik)t},$$

$$\omega = -2k\frac{V_1V_2}{V_1-V_2}, \quad \chi_\pm = k\left(\frac{V_1+V_2}{V_1-V_2}\mp\frac{1}{Q}\right),$$

$$B_\pm = p\left(\frac{V_1+V_2}{V_1-V_2}\mp Q\right), \quad \tan(\beta)=k/(pQ),$$

$$Q = \frac{1}{p}\sqrt{\frac{1}{2}(r+\sqrt{r^2+4k^2p^2})}, \quad r=p^2-k^2-a^2,$$

$$q_n = q(V_{n+1}-V_{n+2}),$$

$$z_n = z+V_nt, \quad n=1,2,3 \mod (3).$$

It is worth noting that the above solution depends upon seven real parameters V_1, V_2, p, k, q, a, and θ. From the definition of Q, it is apparent that these parameters must be chosen in such a way that if $k=0$ then $p^2>a^2$.

The analytical solution (2), while rather complicated at intermediate times, asymptotically consists of one or two coherent structures. In fact, let us consider first the decay process: if we assume $p<0$, for negative large t ($t\to-\infty$) the boomeron is asymptotically composed of two bright pulses (E_1, E_2) and a kinklike pulse (E_3) traveling with the locked velocity V_i. If instead t is large and positive ($t\to+\infty$) the boomeron is composed of two bright pulses (E_1, E_2) and a kinklike pulse (E_3) traveling at the locked velocity V_f ($V_f>V_i$), plus another pulse (E_3) that travels with the linear group velocity V_3. The velocities V_f and V_i can be calculated from (2):

$$V_i = \frac{2V_1V_2}{V_1+V_2-Q(V_1-V_2)}, \tag{3}$$

$$V_f = \frac{2V_1V_2}{V_1+V_2+Q(V_1-V_2)}. \tag{4}$$

The triplet traveling at very large $|t|$ with the locked velocity V_i (V_f) is itself an exact solution of Eqs. (1), namely, it is the unstable (stable) TWRIS as presented in Ref. [15]. Therefore

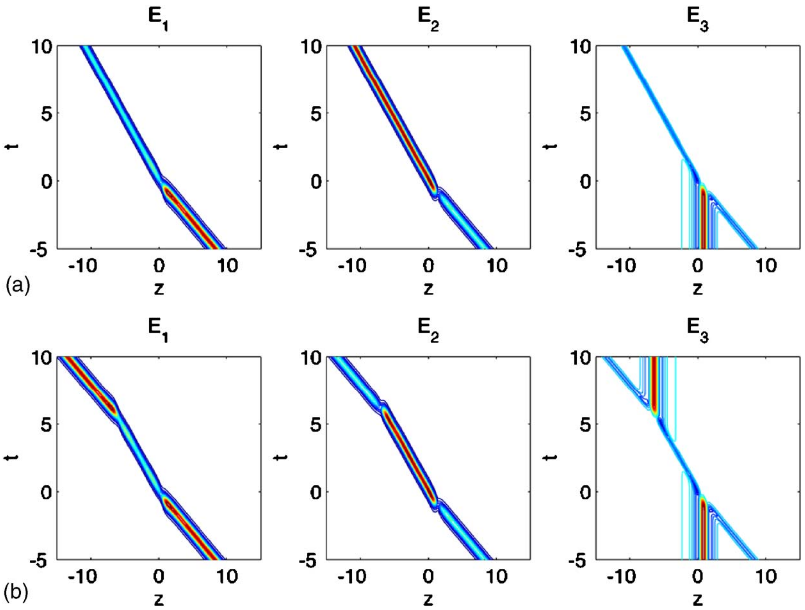

FIG. 2. (Color online) (a) Analytical boomeron solution describing the collision of a stable TWRIS with a single pulse in wave E_3. Parameters are $V_1=2, V_2=1, V_3=0, p=1, a=1, k=0.5, q=1, \theta=\pi/6$. The triplet velocities are $V_i=1.8$ and $V_f=1.1$ ($V_{cr}\approx 1.3$). (b) Numerical double-scattering process.

the boomeron solution (2) provides an exact description of the decay from an unstable to a stable soliton.

Let us consider next the situation where a stable TWRIS collides with an isolated pulse in the wave E_3, namely, excitation by absorption. Once again, this scattering process is exactly described by the boomeron solution (2), and it leads to the excitation of an unstable TWRIS, induced by the absorption of the isolated wave E_3. Indeed, whenever $p>0$ and t is very large and negative, the boomeron (2) is composed of a triplet consisting of two bright pulses (in waves E_1, E_2) and a kinklike pulse (in wave E_3), all traveling with the same velocity V_i, plus an isolated pulse in wave (E_3) that travels with the linear group velocity V_3. The triplet and the isolated pulse collide and, as a result, the pulse in E_3 is completely absorbed by the triplet. Finally, for very large and positive t the boomeron consists of a single triplet formed by two bright pulses (in waves E_1, E_2) and a kinklike pulse (in wave E_3), again traveling together with the velocity V_f ($V_f<V_i$). Note that the asymptotic boomeron triplets traveling with velocities V_i and V_f can be analytically mapped into the stable and unstable TWRISs as given in [15]. In conclusion, the analytical solution (2) with $p>0$ provides the exact description of the excitation of an unstable TWRIS as a result of the inelastic collision between a stable TWRIS and a linear wave packet.

Figure 2(a) displays the analytical boomeron solution corresponding to the collision between a stable TWRIS and a pulse in wave E_3, whereas Fig. 2(b) shows the inelastic scattering of the TWRIS and the linear wave as numerically computed by integrating Eqs. (1) with the initial data at $t=-5$ equal to the solution of Fig. 2(a). As can be seen in Fig. 2(b), the excited unstable TWRIS has a finite lifetime since it eventually decays into a stable or ground state TWRIS via the emission of another linear wave. It is worth noting that both the excitation and the decay processes may be described by properly adjusting the parameters of Eqs. (2).

The dynamics of the scattering between TWRISs and linear waves is analogous to the interaction between radiation and a two-level atom. Indeed, transitions between excited and ground soliton states are induced by the absorption and spontaneous emission of a linear pulse in the wave E_3.

Let us now briefly discuss the role of the various parameters in Eqs. (2). Two of these parameters (i.e., the velocities V_1 and V_2) are fixed by the linear dispersive properties of the medium. We are thus left with five independent real parameters, namely, $p, k, q, a,$and θ (with the restrictions $a>0$ and $0\leq\theta<2\pi$). We point out that our discussion above implies that the specification of these parameters allows one to define the properties of both unstable and stable TWRISs since these solitons result as asymptotic states of the analytic boomeron expression (2) in the limit as $|t|\to\infty$. The parameter p is associated with the rescaling of the wave amplitudes, and of the coordinates z and t, whereas a measures the amplitude of the kink background in wave E_3. The value of k is related to the soliton wave number. The parameter q simply adds a phase shift which is linear in both z and t. Finally, θ fixes the shape of the stationary kink pulse E_3. By adjusting the various degrees of freedom of the boomeron family of solutions (2), one may foresee the dynamical reshaping of the amplitude, phase, and velocity of the TWRIS, as well as fully describe the process of energy exchange among the three waves.

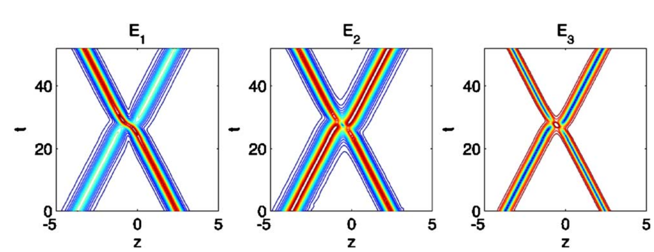

FIG. 3. (Color online) Collision of two stable TWRISs with different velocities. Fast simulton $V=1.9$, slow simulton $V=1.7$. Simulation is performed in a reference frame moving at velocity $V_{ref}=1.8$.

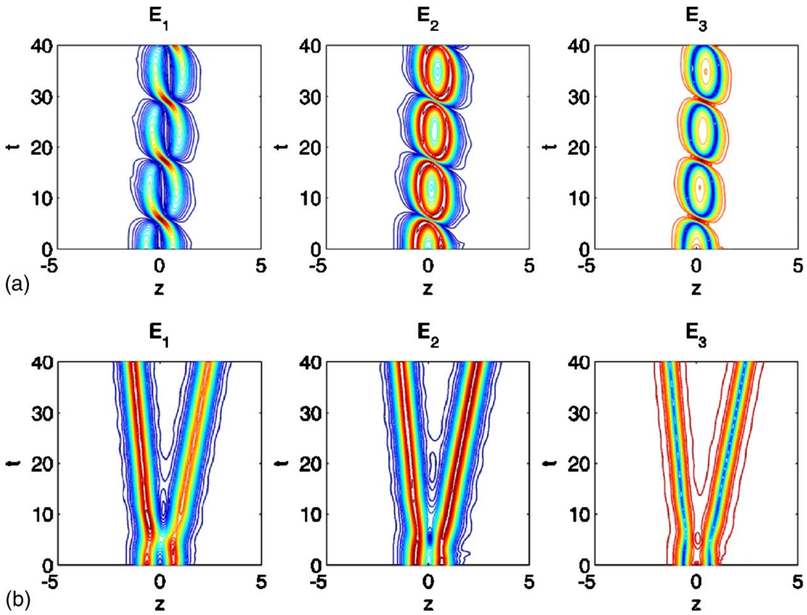

FIG. 4. (Color online) (a) Collision of two equal and in-phase stable TWRISs with the same velocity $V=1.8$; (b) collision of two equal and $\pi/4$ out-of-phase stable TWRISs with the same velocity. Simulations are performed in a reference frame moving at velocity $V_{ref}=1.8$.

In order to emphasize striking features of the scattering between TWRISs and linear waves, let us briefly consider now the collisions between different TWRISs. Since Eqs. (1) are completely integrable, one may intuitively expect (but not take for granted, in view of the previously discussed inelastic soliton absorption and decay phenomena) that interactions between two initially well-separated TWRISs do not modify the shapes of triplets that emerge after the collision. Indeed, the numerical simulation of Fig. 3 shows that this is indeed the case as two TWRISs with different velocities penetrate and cross each other with no change of their shapes. The only effect of the interaction is a spatial shift and a phase shift, as happens with ordinary bright TWRI solitons [12]. However, in a manner similar to cubic nonlinear Schrödinger solitons ([17–19] and references therein), whenever the initial simulton separation is reduced, complex interaction phenomena may take place owing to the excitation of higher-order soliton solutions. For example, Fig. 4(a) shows that two equal and in-phase TWRISs with the same velocity attract each other and periodically collapse, whereas Fig. 4(b) shows that a repulsive force exists between two equal and out-of-phase solitons with the same velocity {the phase difference α between the two solitons is imposed by multiplying the wave E_1 (E_2) of one of the solitons by the phase factor $\exp(i\alpha)$ [$\exp(-i\alpha)$]}. In this case, two distinct TWRISs moving with different velocities emerge from the initial collision. Hence TWRI solitons may cross, attract, or repel each other depending on their initial separation, velocity difference, and relative phase.

In conclusion, we described in terms of analytical solutions the scattering process of three-wave simultons and linear waves. An unstable simulton decays into a stable simulton by accelerating its speed and emitting an isolated pulse. Moreover, a stable triplet may be excited into an unstable simulton by slowing down as a result of the absorption of a linear wave. We revealed the elastic nature of the interactions between stable simultons, finding that simultons with different speeds are stable upon collision, and that interactions of simultons with equal speeds are strongly dependent upon their initial relative phase.

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